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THE STUDY OF THE SOLUTION FOR THE VOLTERRA INTEGRAL EQUATION

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Abstract

In this paper, we will study the solution for Volterra integral equation. Firstly, we prove the existence and uniqueness of the solution for the second Volterra integral equation by using Banach fixed point theorem, then we give a few solving methods for the second Volterra equation. Secondly, we will give the solving methods for the first Volterra equation. Finally, we will discuss the numerical solution for the Volterra equation.

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1. Introduction

In many subjects, such as fluid mechanics, electromagnetic field theory, radiology, geo-physical exploration, automatic control theory, electrical engineering, and microwave technology. Many solutions of those problems can be converted into the corresponding integral equation. For the same question, we can not only solve the problem by using differential equation, but also we can describe it by using integral equation. However, the problem of differential equation itself can be turned into integral equation.

When the problem of differential equation is turned into integral equation, it can reduce the dimension, also weaken the limitation on the unknown functions. Therefore, in order to deal with mathematical problems easier and obtain better results, we use integral form. Especially, many practical differential equation models can be solved by translating into Volterra integral equation. The basic theory [1-10] of Volterra equation was to develop and mature in the 20th century, and gradually applied to physics, mechanics, etc.. So, it is necessary for us to study the solution for Volterra integral equation.

The remainder of this paper is organized as follows. Section 2 briefly describes the prerequisite knowledge and proves the existence and uniqueness of solution for the second type of Volterra integral equation by using Banach fixed point theorem

$$\varphi(x) = \lambda \int_a^x k(x, t)\varphi(t)dt + f(x), \tag{1.1}$$

where $\varphi(x)$ is an unknown function, λ is a parameter, free term $f(x)$ is a square integrable function, i.e., there exists a constant D such that $\int_a^b |f(x)|^2 dx = D^2$. Then introduce a few solving methods. In Section 3, the solving methods were studied for the first type of Volterra integral equation

$$\int_a^x k(x, t)\varphi(t)dt = f(x). \tag{1.2}$$

In Section 4, the numerical solution is listed for Volterra integral equation. Finally, Section 5 concludes this paper.

2. The Solving Problem for the Second Type of Volterra Integral Equation

In order to prove the existence and uniqueness of solutions, we first give some relevant definitions and theorems.

Definition 2.1 ([11]). Assume (X, ρ), $A : X \to X$ is a mapping. If there exists α, $0 \leqslant \alpha < 1$ such that

$$\rho(Ax, Ay) \leqslant \alpha\rho(x, y), \quad \text{for } \forall x, y \in X.$$

Then A is called a compression mapping on X.

Definition 2.2 ([11]). Assume (X, ρ), $\{x_n \subset X\}$. If for any $\epsilon > 0$, there exists a natural number N_ϵ such that

$$\rho(x_n, x_m) < \epsilon, \quad \text{for } \forall m, n > N_\epsilon.$$

Then the sequence $\{x_n\}$ is called Cauchy sequence.

Definition 2.3 (Complete metric space) ([11]). Assume metric space (X, ρ), if any Cauchy sequence in X convergences in (X, ρ), then (X, ρ) is called a complete metric space.

Theorem 2.4 (Banach fixed point theorem) ([11]). *The compression mapping in complete metric space must have a unique fixed point.*

Theorem 2.5 (Promote Banach fixed point theorem) ([11]). *Assume metric space* $\mathbb{R}$ *is complete,* $y = Bx$ *is a mapping which* $\mathbb{R} \to \mathbb{R}$. *If there exists a natural number n such that* B^n *is a compression mapping on* $\mathbb{R}$, *then the mapping B must have a unique fixed point in* $\mathbb{R}$.

2.1. The method of successive approximation to solve the second type of Volterra integral equation and the study about the existence and uniqueness of its solution.

We regard the free term $f(x)$ as zero order approximate solution $\varphi_0(x) = f(x)$, then put $\varphi_0(x)$ substitute into the right of (1.1) and regard the results as one order approximate solution $\varphi_1(x) = f(x) + \lambda\int_a^x k(x, t)\varphi_0(t)dt$, after that put $\varphi_1(x)$ substitute into the right of (1.1), we get $\varphi_2(x) = f(x) + \lambda\int_a^x k(x, t)\varphi_1(t)dt$. And so on, we have the following recurrence relation:

$$\varphi_n(x) = f(x) + \lambda\int_a^x k(x, t)\varphi_{n-1}(t)dt. \tag{2.2.1}$$

By (2.2.1), we have

$$\varphi_1(x) = f(x) + \lambda\int_a^x k(x, t)f(t)dt,$$

$$\varphi_2(x) = f(x) + \lambda\int_a^x k(x, t)f(t)dt + \lambda^2\int_a^x (\int_t^x k(x, u)k(u, t)du)f(t)dt.$$

Let $k_2(x, t) = \int_t^x k(x, u)k(u, t)dt$, then

$$\varphi_2(x) = f(x) + \lambda\int_a^x k(x, t)f(t)dt + \lambda^2\int_a^x k_2(x, t)f(t)dt.$$

And so on, we can get the general expression forms of the approximate solutions $\varphi_n(x)$

$$\varphi_n(x) = f(x) + \sum_{n=1}^{\infty}\lambda^n\int_a^x k_n(x, t)f(t)dt, \tag{2.2.2}$$

where $k_n(x, t) = \int_t^x k(x, u)k_{n-1}(u, t)du$. Here, we can prove the series (2.2.2) convergence for any λ, so the ultimate solution of (1.1) is presented, i.e.,

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^x k_n(x, t) f(t) dt. \tag{2.2.3}$$

In addition, we also assume the solution of (1.1) that can be represented the form of a power series on λ

$$\varphi(x) = \psi_0(x) + \psi_1(x) + \psi_2(x)\lambda^2 + \cdots + \psi_n(x)\lambda^n + \cdots, \tag{2.2.4}$$

then put (2.2.4) substitute into (2.2.2), compare the coefficient of λ on both sides, we get

$$\psi_0(x) = f(x),\ \psi_1(x) = \int_a^x k(x, t)\psi_0(t)dt, \cdots, \psi_n(x) = \int_a^x k(x, t)\psi_{n-1}(t)dt, \tag{2.2.5}$$

thus, (2.2.4) and (2.2.5) are the solutions of equation (1.1). Similarly, we are easy to prove that the series (2.2.4) convergence absolutely and uniformly for any λ, therefore, integral equation (1.1) has a unique solution for any λ. Thus, we have the following theorem:

Theorem 2.6. *Assume* $f(x)$ *is a continuous function in* $[a, b]$, $k(x, t)$ *is continuous in* $\{(x, t) / a \leqslant x \leqslant b, a \leqslant t \leqslant x]\}$ *and* $|k(x, t)| \leqslant M$, *then the integral equation* (1.1) *has a unique solution* $\varphi(x)$, *which is continuous in* $[a, b]$ *for any a constant* λ.

Proof. We consider the mapping $B : \varphi \to B\varphi$, $B\varphi(x) = f(x) + \lambda \int_a^x k(x, t)\varphi(t)dt$ is from $C[a, b] \to C[a, b]$. For any two functions $\varphi_1(x)$ and $\varphi_2(x)$ in $C[a, b]$, when $x \in [a, b]$

$$\begin{aligned} |B\varphi_1(x) - B\varphi_2(x)| &= |\lambda \int_a^x k(x, t)(\varphi_1(t) - \varphi_2(t))dt| \\ &\leqslant |\lambda| M \int_a^x |\varphi_1(t) - \varphi_2(t)| dt \\ &\leqslant |\lambda| M(x - a)\|\varphi_1 - \varphi_2\|. \end{aligned} \tag{2.2.6}$$

Assume, when $n = k$ (2.2.6) is established, i.e.,

$$|B^k\varphi_1(x) - B^k\varphi_2(x)| \leqslant |\lambda|^k M^k \frac{(x-a)^k}{k!}\|\varphi_1 - \varphi_2\|,$$

when $n = k+1$,

$$\begin{aligned}
|B^{(k+1)}\varphi_1(x) - B^{(k+1)}\varphi_2(x)| &= |\lambda\int_a^x k(x, t)(B^k\varphi_1(t) - B^k\varphi_2(t))dt| \\
&\leqslant |\lambda|\int_a^x |k(x, t)|\,|B^k\varphi_1(t) - B^k\varphi_2(t)|dt \\
&\leqslant |\lambda|M\int_a^x |B^k\varphi_1(t) - B^k\varphi_2(t)|dt \\
&\leqslant |\lambda|M\int_a^x |\lambda|^k M^k \frac{(t-a)^k}{k!}\|\varphi_1 - \varphi_2\|dt \\
&= \frac{|\lambda|^{(k+1)}M^{k+1}}{k!}\|\varphi_1 - \varphi_2\|\int_a^x (t-a)^k dt \\
&\leqslant |\lambda|^{(k+1)}M^{k+1}\frac{(x-a)^{(k+1)}}{(k+1)!}\|\varphi_1 - \varphi_2\|.
\end{aligned}$$

Therefore, $|B^n\varphi_1(x) - B^n\varphi_2(x)| \leqslant |\lambda|^n M^n \frac{(x-a)^n}{n!}\|\varphi_1 - \varphi_2\|$. Let natural number n such that

$$\alpha = |\lambda|^n M^n \frac{(x-a)^n}{n!} < 1,$$

then $\|B^n\varphi_1 - B^n\varphi_2\| = \max_{a \leqslant x \leqslant b}|B^n\varphi_1(x) - B^n\varphi_2(x)| \leqslant \alpha\|\varphi_1 - \varphi_2\|$, thus B^n is a compression mapping. So (1.1) has a unique solution $\varphi(x)$, which is continuous in $C[a, b]$ according to Theorem 2.5. □

2.2. The solution for the second type of Volterra integral equation is expressed by solution kernel.

For the second type of Volterra integral equation, we can first calculate overlapping kernel, then get the solution kernel, thus we can get the solution. In the following, we will discuss the solving methods for integral equation of two types kernel.

2.2.1. The integral equation of general type kernel.

By (2.2.5), we have

$$\psi_0(x) = f(x),$$

$$\psi_1(x) = \int_a^x k(x, t)\psi_0(t)dt = \int_a^x k(x, t)f(t)dt,$$

$$\psi_2(x) = \int_a^x k(x, t)\psi_1(t)dt = \int_a^x k(x, t)[\int_a^u k(u, t)f(t)dt]du$$

$$= \int_a^x [\int_t^x k(x, u)k(u, t)du]f(t)dt.$$

Let $k_2(x, t) = \int_t^x k(x, u)k(u, t)du,$ then

$$\psi_2(x) = \int_a^x k_2(x, t)f(t)dt, \cdots, \psi_n(x) = \int_a^x k_n(x, t)f(t)dt, \tag{2.2.7}$$

where

$$k_n(x, t) = \int_t^x k(x, u)k_{n-1}(u, t)dt. \tag{2.2.8}$$

Therefore, we have the following definition:

Definition 2.7. $k_n(x, t) = \int_t^x k(x, u)k_{n-1}(u, t)dt$ is called the n-order overlapping kernel of kernel $k(x, t)$ for the second type of Volterra integral equation.

By (2.2.4) and (2.2.6), we have

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty}\int_a^x k_n(x, t)f(t)dt \cdot \lambda^n = f(x) + \lambda\int_a^x \sum_{n=1}^{\infty}\lambda^{n-1}k_n(x, t)f(t)dt.$$

Let $R(x, t; \lambda) = \sum_{n=1}^{\infty}\lambda^{n-1}k_n(x, t)$, then

$$\varphi(x) = f(x) + \lambda\int_a^x R(x, t; \lambda)f(t)dt. \tag{2.2.9}$$

Equation (2.2.9) is the solution of integral equation (1.1). In the following, we give a definition:

Definition 2.8. $R(x, t; \lambda) = \sum_{n=1}^{\infty}\lambda^{n-1}k_n(x, t)$ is called the solution kernel for the second type of Volterra integral equation.

In conclusion, for the second type of Volterra integral equation, we first solve its overlapping kernel, then get the solution kernel. And finally, the solution of (1.1) can be expressed by using equation (2.2.9). In order to illustrate this solving method, we give an example as follows:

Example 1. Solving the integral equation $\varphi(x) = 2\int_0^x \varphi(t)dt + e^{2x}$.

We first seek the overlapping kernel

$$k_1(x, t) = k(x, t) = 1,$$

$$k_2(x, t) = \int_t^x k(x, u)k_1(u, t)du = \int_t^x du = x - t,$$

$$k_3(x, t) = \int_t^x k(x, u)k_2(u, t)du = \int_t^x (u - t)du = \frac{(x-t)^2}{2},$$

$$\cdots$$

$$k_n(x, t) = \frac{(x-t)^{n-1}}{(n-1)!},$$

then the solution kernel $R(x, t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, t) = \sum_{n=1}^{\infty} \frac{2^{n-1}(x-t)^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \frac{[2(x-t)]^{n-1}}{(n-1)!} = e^{2(x-t)}$, thus the solution of equation $\varphi(x) = 2\int_0^x \varphi(t)dt + e^{2x}$ is

$$\varphi(x) = e^{2x} + 2\int_0^x e^{2(x-t)e^{2t}dt = e^{2x} + 2\int_0^x e^{2x}dt = e^{2x} + 2xe^{2x}}.$$

2.2.2. The integral equation of special type kernel.

For this class integral equation of special type kernel, we will introduce a solving method by Theorems 2.9 and 2.10.

Theorem 2.9 ([12]). *Assume the kernel* $k(x, t)$ *of the second type of Volterra integral equation is* $n-1$ *order polynomial about* $x-t$, *and it has the following form*:

$$k(x, t) = a_0(x) + a_1(x)(x-t) + \frac{a_2(x)}{2!}(x-t)^2 + \cdots + \frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1},$$

where $a_k(x)(k = 0, 1, 2, \cdots, n-1)$ *is continuous in* $[0, a]$. *If* $g(x, t; \lambda)$ *is the solution of differential equation*

$$\frac{d^n g}{dx^n} - \lambda[a_0(x)\frac{d^{n-1}g}{dx^{n-1}} + a_1(x)\frac{d_{n-2}g}{dx^{n-2}} + \cdots + a_{n-1}(x)g] = 0,$$

and it satisfies the conditions as follows:

$$g\,/_{x=t} = \frac{dg}{dx}\,/_{x=t} = \cdots = \frac{d^{n-2}g}{dx^{n-2}}\,/_{x=t} = 0, \quad \frac{d^{n-1}g}{dx^{n-1}}\,/_{x=t} = 1,$$

then the solution kernel

$$R(x, t; \lambda) = \frac{1}{\lambda}\frac{d^n g(x, t; \lambda)}{dx^n}.$$

Theorem 2.10 ([12]). *Assume the kernel* $k(x, t)$ *of the second type of Volterra integral equation is* $n-1$ *order polynomial about* $t-x$, *and it has the following form*:

$$k(x, t) = b_0(t) + b_1(t)(t-x) + \frac{b_2(t)}{2!}(t-x)^2 + \cdots + \frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1},$$

then the solution kernel

$$R(x, t; \lambda) = -\frac{1}{\lambda}\frac{d^n g(t, x; \lambda)}{dt^n},$$

where $g(t, x; \lambda)$ *is the solution of differential equation*

$$\frac{d^n g}{dt^n} - \lambda[b_0(t)\frac{d^{n-1}g}{dt^{n-1}} + b_1(t)\frac{d_{n-2}g}{dt^{n-2}} + \cdots + b_{n-1}(t)g] = 0,$$

and it satisfies the conditions as follows:

$$g\,/_{t=x} = \frac{dg}{dt}\,/_{t=x} = \cdots = \frac{d^{n-2}g}{dt^{n-2}}\,/_{t=x} = 0,\ \frac{d^{n-1}g}{dt^{n-1}}\,/_{t=x} = 1.$$

Similarly, we give an example as follows:

Example 2. Solving the equation $\varphi(x) = \int_0^x (2 + x - t - x^2 - t^2 + 2xt)$ $\varphi(t)dt + f(x)$. Due to the kernel $k(x, t) = 2 + x - t - x^2 - t^2 + 2xt = 2 + (x-t) - (x-t)^2$, so $a_0(x) = 2$, $a_1(x) = 1$, $a_2(x) = 2$, $n = 3$, thus the solution of differential equation

$$\frac{d^3 g(x, t; 1)}{dx^3} - [2\frac{d^2 g(x, t; 1)}{dx^2} + \frac{dg(x, t; 1)}{dx} - 2g(x, t; 1)] = 0,$$

is

$$g(x, t; 1) = c_1(t)e^{-x} + c_2(t)e^{x} + c_3(t)e^{2x},$$

meanwhile, we can get $c_1(t) = \frac{1}{6}e^t$, $c_2(t) = -\frac{1}{2}e^{-t}$, $c_3(t) = \frac{1}{3}e^{-2t}$ by the following conditions:

$$g\,/_{x=t} = \frac{dg(x,\,t;\,1)}{dx}\,/_{x=t} = 0, \quad \frac{d^2g(x,\,t;\,1)}{dx^2}\,/_{x=t} = 1,$$

then

$$g(x,\,t;\,1) = \frac{1}{6}e^{t-x} - \frac{1}{2}e^{x-t} + \frac{1}{3}e^{2(x-t)},$$

thus $R(x,\,t;\,1) = \dfrac{d^2g}{dx^2} = \dfrac{1}{6}e^{t-x} - \dfrac{1}{2}e^{x-t} + \dfrac{4}{3}e^{2(x-t)}$. So the solution is

$$\varphi(x) = \int_0^x \left(\frac{1}{6}e^{t-x} - \frac{1}{2}e^{x-t} + \frac{4}{3}e^{2(x-t)}\right)f(t)dt + f(x).$$

2.3. Transforming the second type of Volterra integral equation into the Cauchy problem of ordinary differential equation.

If the kernel $k(x,\,t)$ and free term $f(x)$ of the second Volterra integral equation (1.1) have continuous derivatives $k'_x(x,\,t)$ and $f'(x)$, then we obtain the derivative results (one or more) for the integral equation (1.1). In many cases, we can transform the integral equation (1.1) into the Cauchy problem of ordinary differential equation.

Example 3. Solving the integral equation $\varphi(x) = 2e^x + 4x - 2\int_0^x (x-t)\varphi(t)dt$. We can get $\varphi'(x) = 2e^x + 4 - 2\int_0^x \varphi(t)dt$, $\varphi''(x) = 2e^x - 2\varphi(x)$ by using derivative, and $\varphi(0) = 2$, $\varphi'(0) = 6$. So, the solution is $\varphi(x) = \dfrac{4}{3}\cos\sqrt{2}x + \dfrac{8\sqrt{2}}{3}\sin\sqrt{2}x + \dfrac{2}{3}e^x$ by solving the above Cauchy problem.

If the kernel of (1.1) is degenerate kernel, then for the second type of Volterra integral equation (i.e., (2.2.10)), we have the following solving method:

$$\varphi(x) = f(x) + \int_a^x \left[\sum_{i=1}^n a_i(x)b_i(t)\right]\varphi(t)dt. \tag{2.2.10}$$

Let $\varphi_1(x) = \int_a^x b_1(t)\varphi(t)dt, \cdots, \varphi_n(x) = \int_a^x b_n(t)\varphi(t)dt,$ then the solution of (2.2.10) can be represented as follows:

$$\varphi(x) = f(x) + \sum_{i=1}^{n} a_i(x)\varphi_i(x). \tag{2.2.11}$$

Here, our aim is to solve $\varphi_i(x)$. We seek every derivative of $\varphi_1(x), \cdots,$ $\varphi_n(x),$ and obtain the following equations:

$$\varphi_1'(x) = b_1(x)f(x) + \sum_{i=1}^{n} b_1(x)a_i(x)\varphi_i(x),$$

$$\cdots \qquad \cdots$$

$$\varphi_n'(x) = b_n(x)f(x) + \sum_{i=n}^{n} b_n(x)a_i(x)\varphi_i(x),$$

and $\varphi_1(a) = \varphi_2(a) = \cdots = \varphi_n(a) = 0.$ Then we can get $\varphi_i(x), i = 1, 2, \cdots, n$ by solving the above equations with initial conditions. Finally, we obtain the solution of (2.2.10) by putting $\varphi_i(x)$ substituted into (2.2.11).

Example 4. Solving the integral equation with degenerate kernel $\varphi(x) = e^x + \int_0^x e^{x-t}\varphi(t)dt.$ Due to $\varphi(x) = e^x + e^x\int_0^x e^t\varphi(t)dt,$ let $\int_0^x e^{-t}\varphi(t)dt = \varphi_1(x),$ then, we have

$$\varphi(x) = e^x + e^x\varphi_1(x), \tag{2.2.12}$$

and $\varphi_1'(x) = e^{-x}\varphi(x),$ but from (2.2.12), we have $\varphi_1'(x) = e^{-x}(e^x + e^x\varphi_1(x)),$ thus $\varphi_1'(x) - \varphi_1(x) = 1,$ also $\varphi_1(0) = 0,$ so we can obtain $\varphi_1(x) = e^x - 1$ by solving the problem of differential equation with initial condition

$$\varphi_1'(x) - \varphi_i(x) = 1,$$

$$\varphi_1(0) = 0.$$

Therefore, the solution is $\varphi(x) = e^x\varphi_1'(x) = e^x(e^x) = e^{2x}.$

3. The Solving Problem for the First Type of Volterra Integral Equation

For the Volterra integral equation (1.2), we can transform it into the second type of Volterra integral equation, then solving it with the help of the methods that introduced in Section 2. There are two methods (derivative method and subsection integration method) will be discussed in this section.

3.1. Derivative method

We first give a theorem of derivative method about transforming the first type of Volterra integral equation into the second type of Volterra integral equation.

Theorem 3.1 ([12]). *For the first type of Volterra integral equation* (1.2), *if* $k(x, t)$ *and* $f(x)$ *are derivability,* $k(x, t) \neq 0$, $(a \leqslant x \leqslant b)$, $k(x, x)$ *and* $\frac{\partial k(x, t)}{\partial x}$ *are continuous in* $[a, b]$ *and* $a \leqslant t \leqslant x \leqslant b$, $f(a) = 0$, *then the equation* (1.2) *is equivalent to the second type of Volterra integral equation* (3.1).

$$\varphi(x) + \int_a^x \frac{k'_x(x, t)}{k(x, x)} \varphi(t) dt = \frac{f'(x)}{k(x, x)}. \tag{3.1}$$

In the following, we will give an example:

Example 5. Solving the first Volterra integral equation

$$\int_0^x \sinh(x - t)\varphi(t) dt = x^3 e^{-x}. \tag{3.2}$$

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[1]Notice: The first type of Volterra integral equation has solution only if the free term $f(x)$ satisfies the compatibility condition. Viz., for any kernel $k(x, t)$, a necessary condition for the existence at $x = a$ of its solution is $f(a) = 0$. Therefore, the first type of Volterra integral equation must satisfy the compatibility condition before every time derivative, then the solving for the second type of Volterra integral equation that is transformed into by the first type of Volterra integral equation can be meaningful.

We take a derivative about x on both sides of the equation (3.2), then we obtain

$$\cosh(x-t)\int_0^x \varphi(t)dt = 3x^2e^{-x} - x^3e^{-x}, \tag{3.3}$$

we can see that the equation (3.3) is still the first type of Volterra integral equation, so we again take a derivative about x on both sides of equation (3.3), then we get

$$\varphi(x) + \int_0^x \sinh(x-t)\varphi(t)dt = 6xe^{-x} - 6x^2e^{-x} + x^3e^{-x},$$

thus $\varphi(x) = 6xe^{-x} - 6x^2e^{-x}$. Due to the equations (3.2) and (3.3) are all satisfy the compatibility condition, so the solution of equation (3.2) is $\varphi(x) = 6xe^{-x} - 6x^2e^{-x}$.

3.2. Subsection integration method

The equation (1.2) also can be transformed into the second type of Volterra integral equation by using subsection integration method. For the integral on the left side of the equation (1.2), let

$$\psi(x) = \int_a^x \varphi(t)dt, \tag{3.4}$$

we have $k(x, x)\psi(x) - \int_a^x \frac{\partial k(x, t)}{\partial t}\psi(t)dt = f(x)$, if $k(x, x) \neq 0$, then

$$\psi(x) - \int_a^x [\frac{\partial k(x, t)}{\partial t} / k(x, x)]\psi(t)dt = \frac{f(x)}{k(x, x)}, \tag{3.5}$$

when $\frac{f(x)}{k(x, x)}$ and the kernel are continuous, the equation (3.5) has only solution $\psi(x)$, and $\phi(x) = \psi'(x)$ is the solution that we wanted by the equation (3.4).

Example 6. Solving the first type of Volterra integral equation $\int_0^x (1 - x^2 + t^2)\varphi(t)dt = \frac{x^2}{2}$. According to the above mentioned method, we let

$$\psi(x) = \int_0^x \varphi(t)dt,$$

then

$$\int_0^x (1 - x^2 + t^2)\varphi(t)dt = \frac{x^2}{2},$$

can be transformed into

$$\int_0^x (1 - x^2 + t^2)d\psi(t) = \frac{x^2}{2},$$

thus

$$(1 - x^2 + t^2)\psi(t)/_0^x - \int_0^x \psi(t)d(1 - x^2 + t^2) = \frac{x^2}{2},$$

then we have

$$\psi(x) - 2\int_0^x t\psi(t)dt = \frac{x^2}{2}.$$

Therefore, the original equation can be transformed into the Cauchy problem of ordinary differential equation

$$\psi'(x) - 2x\psi(x) = x,$$

$$\psi(x) = 0,$$

so we can get $\psi(x) = -\frac{1}{2} + \frac{1}{2}e^{x^2}$ by solving the above Cauchy problem. Finally, the solution is $\varphi(x) = \psi'(x) = \frac{1}{2}e^{x^2} \cdot 2x = xe^{x^2}$.

4. The Numerical Solution of Volterra Integral Equation

In practical application, we are unable or difficult to work out the accurate analytical solution for most of the Volterra integral equations. But in most cases, we can work out its approximate solution. In this section, we mainly introduce a kind of method (called numerical integral method [12]) that is used to work out the approximate solution of Volterra integral equation.

4.1. The numerical solution for the second type of Volterra integral equation.

For equation (1.1), let $x = x_j$, $(j = 1, 2, \cdots, n)$, we use finite sum to replace the integral of (1.1), and then obtain

$$\varphi_j - \lambda \sum_{m=1}^{j} A_m k_{jm} \varphi_m = f_j,\ j = 1, 2, \cdots, n, \tag{4.1}$$

where $\varphi_j = \varphi(x_j)$, $k_{jm} = k(x_j, x_m)$, $f_j = f(x_j)$.

The formula (4.1) is a system of linear equations. Due to the coefficients of (4.1) can construct a lower triangular matrix, so $\varphi_1, \varphi_2, \cdots, \varphi_n$ can be solved easily. Therefore, the numerical solution of equation (1.1) can be expressed as follows:

$$\varphi(x) \approx \lambda \sum_{m=1}^{j} A_m k(x, x_m) \varphi_m + f(x),\ x_{j-1} < x \leqslant x_j, \tag{4.2}$$

when $j \to \infty$, equation (4.2) convergences to the analytical solution of equation (1.1) uniformly. In the following, we give an example to illustrate how to use this method.

Example 7. Seeking the numerical solution for the second type of Volterra integral equation $\varphi(x) = 2\int_0^x \varphi(t)dt + e^{2x}$ in $[0, 1]$, the analytical solution of which is $\varphi(x) = e^{2x} + 2xe^{2x}$.

(1) Firstly, we use the trapezoid formula of $n = 6$ to seek the numerical solution. In this moment, $h = 0.2$, then $A_1 = A_6 = \frac{h}{2} = 0.1$, $A_m = h = 0.2$, $m = 2, 3, 4, 5$. Let $x = x_j$, $(j = 1, \cdots, 6)$, we can obtain

$$\varphi_1 = f_1,$$

$$\varphi_j - 2\int_0^{x_j} \varphi(t)dt = e^{2x_j}, \quad j = 2, \cdots, 6. \tag{4.3}$$

Then, we use finite sun to replace the integral section of equation (4.3) and get

$$\varphi_j - \lambda \sum_{m=1}^{j} A_m k_{jm} \varphi_m = f_j, \quad j = 1, 2, \cdots, 6, \tag{4.4}$$

finally, solving the system of linear equations (4.4), we have

$\varphi(x = 0) = 1$, $\varphi(x = 0.2) = 2.114780872$, $\varphi(x = 0.4) = 4.089316597$,

$\varphi(x = 0.6) = 7.502194888$, $\varphi(x = 0.8) = 13.29443671$, $\varphi(x = 1) = 22.98668466$.

(2) Secondly, we use the trapezoid formula of $n = 11$ to seek the numerical solution. In this moment, $h = 0.1$, then $A_1 = A_{11} = \frac{h}{2} = 0.05$, $A_m = h = 0.1$, $m = 2, \cdots, 10$. Similarly, we have

$\varphi(x_1 = 0) = 1$, $\varphi(x_2 = 0.2) = 2.094966394$, $\varphi(x_3 = 0.4) = 4.026312056$,

$\varphi(x_4 = 0.6) = 7.352474275$, $\varphi(x_5 = 0.8) = 12.97917831$,

$\varphi(x_6 = 1) = 22.36611733$.

Due to the analytical solution is $\varphi(x) = e^{2x} + 2xe^{2x}$, so we have

$\varphi(0) = 1$, $\varphi(0.2) = 2.088554577$, $\varphi(0.4) = 4.005973671$,

$\varphi(0.6) = 7.30425723$, $\varphi(0.8) = 12.8778843$, $\varphi(1) = 22.1671683$.

The results of Example 7 (under different step sizes) are listed in Table 1.

Table 1. Numerical solutions of different step sizes for the second type of Vollterra integral equation

Step size h/x	0	0.2	0.4	0.6	0.8	1.0
0.2	1	2.114780872	4.089316597	7.502194888	13.29443671	22.98668466
0.1	1	2.094966394	4.026312056	7.352474275	12.97917831	22.36611733
analytical solution	1	2.088554577	4.005973671	7.304257230	12.87788430	22.16716830

4.2. The numerical solution for the first type of Volterra integral equation.

For the numerical solution of the first type of Volterra integral equation, it is not necessary for us to transform it to the second type of Volterra integral equation. We can seek the numerical solution directly. In the following, we will give an example by using the similar method that introduced in Subsection 4.1.

Example 8. Seeking the numerical solution for the first type of Volterra integral equation $\int_0^x (1 - x^2 + t^2)\varphi(t)dt = \frac{x^2}{2}$ in $[0, 1]$, the analytical solution of which is $\varphi(x) = xe^{x^2}$.

(1) Firstly, when the step size $h = 0.2$, for the above integral equation,

$$k(x, t) = 1 - x^2 + t^2,\ f(x) = \frac{x^2}{2}.$$

We use finite sum to replace the integral section of $\int_0^x (1 - x^2 + t^2)\varphi(t)$ $dt = \frac{x^2}{2}$, we have

$$\lambda \sum_{m=1}^{j} A_m k_{jm} \varphi_m = f_j, \quad j = 2, 3, \cdots, 6, \tag{4.5}$$

where $A_1 = A_j = \frac{h}{2}$, $j = 2, \cdots, 6$, $A_{j-1} = h$, $j = 3, \cdots, 6$, $\varphi_m = \varphi(x_m)$, $k_{jm} = k(x_j, x_m)$, $f_j = f(x_j)$. When $k(x, x) \neq 0$, $\varphi(0) = \frac{f'(x_1)}{k(x_1, x_1)} = 0$. Thus, solving the system of linear equations (4.5), we can obtain

$$\varphi(x = 0) = 0, \quad \varphi(x = 0.2) = 0.2, \quad \varphi(x = 0.4) = 0.448,$$

$$\varphi(x = 0.6) = 0.8112, \quad \varphi(x = 0.8) = 1.405952, \quad \varphi(x = 1) = 2.45695744.$$

(2) Secondly, when the step size $h = 0.1$, according to the above process (1), we have

$$\varphi(x_1 = 0) = 0, \quad \varphi(x_2 = 0.2) = 0.206, \quad \varphi(x_3 = 0.4) = 0.463684,$$

$$\varphi(x_4 = 0.6) = 0.8467785264, \quad \varphi(x_5 = 0.8) = 1.486518226,$$

$$\varphi(x_6 = 1) = 2.644676007.$$

Due to the analytical solution is $\varphi(x) = xe^{x^2}$, so we have

$$\varphi(0) = 0, \quad \varphi(0.2) = 0.2081621548, \quad \varphi(0.4) = 0.4694043484,$$

$$\varphi(0.6) = 0.8599976487, \quad \varphi(0.8) = 1.517184703, \quad \varphi(1) = 2.718281828.$$

The results of Example 8 (under different step sizes) are listed in Table 2.

Table 2. Numerical solutions of different step sizes for the first type of Vollterra integral equation

Step size h / x	0	0.2	0.4	0.6	0.8	1.0
0.2	0	0.200000000	0.448000000	0.811200000	1.405952000	2.456967440
0.1	0	0.206000000	0.463684000	0.846778530	1.486518230	2.644676007
analytical solution	0	0.208162155	0.469404350	0.859997650	1.517184700	2.718281828

5. Conclusion

In this paper, we mainly discussed the existence and uniqueness of the solution for Volterra integral equation, and give some kinds of solving methods, then verified the solving methods that we have given by examples. But for most of Volterra integral equation, we cannot seek its analytical solution, therefore, we discussed a numerical method (numerical integral method) that is used to seek the numerical solution, and gave examples to illustrate the effectiveness of numerical integral method.

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